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LETTER TO THE EDITOR

Random-site spin-glass models

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Abstract. A simple general method is presented for solving mean-field spin-glass models where the bond randomness is expressible in terms of an underlying site randomness. Both separable and non-separable models can be solved.

The mean-field models of spin glasses discussed in the literature can be divided into two classes, the first class consisting of the true random-bond models where the couplings between interacting spins are taken to be independent random variables (Sherrington and Kirkpatrick 1975, Derrida 1981), and the second class containing those models where the bond randomness is expressible in terms of some underlying hidden site randomness (e.g. Luttinger 1976, van Hemmen 1982). The bond randomness in models of the second class is in general not uncorrelated, even if the underlying site randomness is. It has been pointed out by Benamira *et al* (1985) that this feature retains an important physical aspect of true spin glasses, viz that they are random with respect to the positions of magnetic impurities.

Known models in the 'random-site class' share the important feature that the random part of the interaction is a *bilinear* function of the underlying site randomness, and hence can quite generally be expressed as (see Benamira *et al* 1985)

$$J_{ij} = N^{-1}(\boldsymbol{\xi}_i, \mathbb{J}\boldsymbol{\xi}_j) \tag{1}$$

where the $\boldsymbol{\xi}$ are stochastic vectors in $\mathbb{R}^p$, $\mathbb{J}$ a real symmetric $p \times p$ matrix and N denotes the number of spins in the system. Equation (1) includes, for instance, the models of Mattis (1976), Luttinger (1976) and van Hemmen (1982) and their generalisations introduced by Provost and Vallée (1983). While these so-called separable models are capable of reproducing certain thermodynamic properties of the spin-glass phase, such as the plateau in the DC susceptibility (van Hemmen *et al* 1983), they invariably lack a major feature of spin glasses, namely the existence of a large number of metastable low-temperature phases (Choy and Sherrington 1984). In a recent paper, however, Benamira *et al* (1985) showed how this deficiency may be overcome in the framework of separable models, but only at the cost of introducing an infinite number ($p \to \infty$) of random variables per lattice site.

In this letter we take a different approach to the solution of mean-field spin-glass models in the random-site class. Utilising discrete probability distributions, we analyse and solve spin-glass models of a novel type, where the random part of the interaction is given by

$$J_{ij} = (J/N) f(\boldsymbol{\xi}_i; \boldsymbol{\xi}_j). \tag{2}$$

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As before the $\boldsymbol{\xi}$ are taken to be stochastic vectors in $\mathbb{R}^p$ with some common distribution, but $f(\boldsymbol{\xi}; \boldsymbol{\xi}')$ is now an arbitrary real symmetric function and no longer needs to be a bilinear form.

Our method of solution is rather elementary and requires no replicas. It is based on the observation that every Hamiltonian of the random-site class

$$H_N = -(J_0/N)\sum_{(ij)}\sigma_i\sigma_j - \sum_{(ij)} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i \tag{3}$$

with J_{ij} given by equation (2), can be expressed as a quadratic form of magnetisations of certain sublattices which are of macroscopic size, if the probability distributions of the $\boldsymbol{\xi}$ are suitably chosen.

To be specific, we assume the components of the p-dimensional vectors $\boldsymbol{\xi}_i$ in equations (1) and (2) to be independent random variables drawn from the set $A_n = \{a_1, a_2, \ldots, a_n\}$ with uniform probability. The generalisation to non-uniform distributions is trivial, as indicated below. The elements of A_n must, of course, be disposed of in a way which depends on the desired probability distribution for the J_{ij}.

Every quenched configuration of the random vectors $\boldsymbol{\xi}_i$ leads to a partition of the lattice Ω_N into n^p disjoint sublattices and the Hamiltonian, therefore, depends on a given configuration $\{\boldsymbol{\xi}_i\}$ only through this partitioning, so that the sum over all spin configurations can be performed in an essentially trivial way. Indeed, given p and A_n, the $\boldsymbol{\xi}$ can only be drawn from a finite set A of n^p different vectors. Introducing a single index γ to enumerate the n^p vectors $\boldsymbol{a}_\gamma$ in A, we find that the sublattices

$$\Omega_\gamma = \{i \in \Omega_N;\ \boldsymbol{\xi}_i = \boldsymbol{a}_\gamma\} \qquad \boldsymbol{a}_\gamma \in A \tag{4}$$

are disjoint and together make up the whole lattice. If we introduce n^p corresponding block spins or sublattice magnetisations

$$M_\gamma = \sum_{i\in\Omega_\gamma}\sigma_i \qquad \gamma = 1, \ldots, n^p \tag{5}$$

the Hamiltonian takes a particularly compact form in terms of the M_γ:

$$-\beta H_N = K/(2N)\sum_{\gamma\gamma'}^{n^p} \mathbf{V}_{\gamma\gamma'}M_\gamma M_{\gamma'} + H\sum_{\gamma}^{n^p} M_\gamma. \tag{6}$$

Here we have defined

$$\mathbf{V}_{\gamma\gamma'} = \alpha + f(\boldsymbol{a}_\gamma; \boldsymbol{a}_{\gamma'}) \tag{7}$$

and

$$K = \beta J \qquad H = \beta h \qquad \alpha = J_0/J \tag{8}$$

and have omitted irrelevant terms of order unity, which do not contribute to the free energy density in the thermodynamic limit.

Since $\mathbf{V}$ is a symmetric matrix, we can reduce it to a diagonal form by an orthogonal transformation

$$\mathbf{D} = \mathbf{Q}^{\mathrm{T}}\mathbf{V}\mathbf{Q} \qquad \mathbf{D}_{\gamma\gamma'} = \lambda_\gamma\delta_{\gamma\gamma'} \tag{9}$$

so that

$$-\beta H_N = K/(2N)\sum_{\mu}{}' \lambda_\mu x_\mu^2 + H\sum_\gamma M_\gamma \tag{10}$$

with

$$x_\mu = \sum_\gamma M_\gamma \mathbf{Q}_{\gamma\mu}. \tag{11}$$

The primed summation in equation (10) indicates that non-contributing terms with $\lambda_\mu = 0$ have been omitted. To evaluate the partition function

$$Z_N = \mathrm{Tr}_{\{\sigma\}} \exp[-\beta H_N(\{M_\gamma\})] \tag{12}$$

we use the Hubbard-Stratonovich transformation (Hubbard 1959) and obtain

$$Z_N(\{\omega_\gamma\}) = \prod_\mu{}' \{2\pi K|\lambda_\mu|/N\}^{-1/2} \times \int_{-\infty}^{+\infty} \prod_\mu{}' \mathrm{d}t_\mu \exp\left[-N\left(1/(2K)\sum_\mu{}' t_\mu^2/|\lambda_\mu| - \sum_\gamma \omega_\gamma \ln(2\cosh\alpha_\gamma)\right)\right] \tag{13}$$

where

$$\alpha_\gamma = H + \sum_\mu{}' \mathbf{Q}_{\gamma\mu} s_\mu t_\mu \qquad s_\mu^2 = \mathrm{sgn}(\lambda_\mu) \tag{14}$$

and where

$$\omega_\gamma = |\Omega_\gamma|/N \tag{15}$$

denotes the fraction of lattice sites which belong to the sublattice Ω_γ. According to the strong law of large numbers (Rényi 1971), the quantities ω_γ converge with probability one to their mean value n^{-p} in the thermodynamic limit. In the case of non-uniform probability distributions the ω_γ would simply converge to some p_γ according to their probability.

The integrals in equation (13) are evaluated by the method of steepest descents, and the thermodynamic limit of the quenched free energy density is found to be

$$-\beta f = \max_{\{t_\mu\}}\left(-(2K)^{-1}\sum_\mu{}' t_\mu^2\left\{|\lambda_\mu| + n^{-p}\sum_\gamma \ln\left[2\cosh\left(H + \sum_\mu{}' \mathbf{Q}_{\gamma\mu} s_\mu t_\mu\right)\right]\right\}^{-1}\right). \tag{16}$$

The maximising parameters t_μ are among the solutions of the following set of transcendental equations:

$$t_\mu/(K|\lambda_\mu|) = n^{-p} s_\mu \sum_\gamma \mathbf{Q}_{\gamma\mu} \tanh\left(H + \sum_\nu{}' \mathbf{Q}_{\gamma\nu} s_\nu t_\nu\right) \tag{17}$$

and again we note that only those t_μ are involved which correspond to non-zero eigenvalues λ_μ.

It is easy to show that the t_μ as solutions of equations (17) are linear combinations of sublattice magnetisation densities defined by

$$m_\gamma = \tanh\left(H + \sum_\nu{}' \mathbf{Q}_{\gamma\nu} s_\nu t_\nu\right). \tag{18}$$

Since sublattice magnetisations are real quantities, it follows from equations (17) that those t_μ which correspond to negative eigenvalues, hence having $s_\mu^2 = -1$, must be purely imaginary. It is therefore convenient to rewrite the fixed point equations (17) in terms of the real parameters

$$y_\mu = s_\mu t_\mu \tag{19}$$

which leads to

$$y_\mu = K\lambda_\mu n^{-p} \sum_{\gamma=1}^{n^p} \mathbf{Q}_{\gamma\mu} \tanh\left(H + \sum_\nu{}' \mathbf{Q}_{\gamma\nu} y_\nu\right). \tag{20}$$

In terms of the solution $\{y_\mu\}$ of the above equations which corresponds to the maximum in equation (16), the free energy per spin is simply given by

$$-\beta f = n^{-p} \sum_\gamma \ln\left[2\cosh\left(H + \sum_\nu{}' \mathbf{Q}_{\gamma\nu} y_\nu\right)\right] - (2K)^{-1} \sum_\mu{}' y_\mu^2/\lambda_\mu. \tag{21}$$

Whenever equation (20) allow several solutions, one still has, of course, to decide which of the phases (solutions) minimises the free energy, and hence is absolutely stable, and which of them must be regarded as metastable (corresponding to local minima of the free energy) or even unstable.

The most important consequence of the above results is that the number of order parameters necessary to describe the system is equal to the number of non-zero eigenvalues of the matrix **V**. Since the dimension of **V** is n^p, where n denotes the number of elements of A_n, one would expect the rank of **V** to be, to a large extent, at our disposal and that it would ultimately increase with n. There is in this respect, however, a fundamental difference between separable and non-separable models in the random-site class. While for non-separable models the rank of **V** does indeed increase with n, this proves not to be the case for separable models. This is easily seen by noting that for separable models (cf equation (1)) the matrix **V** is a sum of p rank-1 dyadic matrices so that its rank is at the most p, irrespective of the dimension of **V**, and hence independent of the probability distribution for the J_{ij}. This is, of course, related to the fact that separable models can be solved in terms of their p natural order parameters (Provost and Vallée 1983).

It should perhaps be noted that it is easy and sometimes advantageous to eliminate the diagonalising matrix **Q** from the final results (equations (20) and (21)) so as to formulate the free energy and the associated set of fixed point equations in terms of the sublattice magnetisations m_γ and the 'interaction matrix' **V**. The number of fixed point equations, however, will then increase from rank(**V**) to dim(**V**).

In conclusion, we have presented a simple method for solving mean-field spin-glass models in the random-site class. The methods available for solving random-site class models are invariably restricted to the *separable* models, since they utilise, in one way or another, the fact that the random part of the interaction is a *bilinear function of the site randomness*, whereas our method is free of such a restriction, requiring nothing but *bilinearity in the spins*. This is ultimately facilitated by the different nature of our order parameters, which are sublattice magnetisations or suitable linear combinations thereof. As it stands, our method is applicable only to *discrete* probability distributions. Continuous distributions can be handled, but this requires further ingredients and a somewhat different approach (van Hemmen *et al* 1986). Nevertheless the class of exactly soluble models has been considerably extended.

Thinking of possible applications of the present approach, the strong influence of probability distributions in non-separable models may help in adapting spin-glass models to experimental facts, such as reentrance phenomena or the correspondence known to exist between atomic ordering and the magnetic phase diagram. Details will be presented elsewhere (Grensing and Kühn 1986). Other fields of application include, e.g., neural networks with non-linear synapses (van Hemmen and Kühn 1986).

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